Final exam for Kwantumfysica 1 - 2009-2010
Thursday 21 January 2010, 8:30 - 11:30

READ THIS FIRST:

- Note that the lower half of this page lists some useful formulas and constants.
- Clearly write your name and study number on each answer sheet that you use.
- On the first answer sheet, write clearly the total number of answer sheets that you turn in.
- Note that this exam has 3 questions, it continues on the backside of the papers!
- Start each question (number 1, 2, 3) on a new answer sheet.
- The exam is open book within limits. You are allowed to use the book by Liboff, the handout *Extra note on two-level systems and exchange degeneracy for identical particles*, and one A4 sheet with notes, but nothing more than this.
- If it says "make a rough estimate", there is no need to make a detailed calculation, and making a simple estimate is good enough. If it says "calculate" or "derive", you are supposed to present a full analytical calculation.
- If you get stuck on some part of a problem for a long time, it may be wise to skip it and try the next part of a problem first.
- If you are ready with the exam, please fill in the **course-evaluation question sheet**. You can keep working on the exam until 11:30, and fill it in shortly after 11:30 if you like.

Useful formulas and constants:

Electron mass	m_e	$= 9.1 \cdot 10^{-31}$ kg
Electron charge	$-e$	$= -1.6 \cdot 10^{-19}$ C
Planck's constant	h	$= 6.626 \cdot 10^{-34}$ Js $= 4.136 \cdot 10^{-15}$ eVs
Planck's reduced constant	$\hbar$	$= 1.055 \cdot 10^{-34}$ Js $= 6.582 \cdot 10^{-16}$ eVs

Fourier relation between x-representation and k-representation of a state

$$\Psi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \overline{\Psi}(k) e^{ikx}\, dk$$

$$\overline{\Psi}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Psi(x) e^{-ikx}\, dx$$

Problem 1

Consider a one-dimensional (with position x) particle-in-the-box system, where a quantum particle is in an infinitely deep potential well with $V = 0$ for $|x| < a/2$, and $V = \infty$ elsewhere ($a = 1$ nm is the width of the well). The particle has a mass $m = 2 \cdot 10^{-30}$ kg. The particle is brought into the box with a mechanism that results in a wavefunction for the particle that is evenly distributed in the well, $\Psi(x) = e^{i\varphi}/\sqrt{a}$ for $|x| < a/2$ (where i the imaginary number, and $\varphi = \frac{1}{5}\pi$ the phase of the state) and zero elsewhere.

a) Represent this state in the k-representation (a wavefunction that is a function of wave number k).

b) What is the expectation value $\langle v \rangle$ for the velocity v of the particle?

c) In an experiment, the velocity of the particle is measured many times. Before each measurement the state of the particle is first again prepared in the state $\Psi(x) = e^{i\varphi}/\sqrt{a}$. The results show that 90% of the measured values v_m are in the interval $\langle v \rangle - \Delta v_{90\%} < v_m < \langle v \rangle + \Delta v_{90\%}$. Make a rough estimate for the value of $\Delta v_{90\%}$ (give a real number in units of m/s).

d) Show that the state $\Psi(x) = e^{i\varphi}/\sqrt{a}$ is not an energy eigenstate of the system.

e) In Dirac notation, the state $\Psi(x) = e^{i\varphi}/\sqrt{a}$ is represented as $|\Psi\rangle$. Prove the relation $\Psi(x) = \langle x|\Psi\rangle$ ($|x\rangle$ is the eigenvector with eigenvalue x for the position operator $\hat{x}$).

f) $\Psi(x) = e^{i\varphi}/\sqrt{a}$ and $|\Psi\rangle$ as mentioned in **e)** do represented the same physical state. Explain why it is incorrect to write down $\Psi(x) = |\Psi\rangle$ when one aims to express that it concerns the same state in different representations.

g) Since the state $|\Psi\rangle$ is not an energy eigenstate of the system, it must be a superposition of energy eigenstates, which can be represented as $|\Psi\rangle = \sum_n c_n|\varphi_n\rangle$, (where $|\varphi_n\rangle$ the energy eigenstate that is associated with energy eigenvalue E_n). Proof the relation $c_n = \langle \varphi_n|\Psi\rangle$. Use Dirac notation.

h) What is the value of c_n for the case $|\varphi_n\rangle = |\varphi_1\rangle$ (which is $\varphi_1(x) = \sqrt{\frac{2}{a}}\cos(\frac{\pi x}{a})$ for $|x| < a/2$ and zero elsewhere)? And what is c_n for the case $|\varphi_n\rangle = |\varphi_2\rangle$ (which is $\varphi_2(x) = \sqrt{\frac{2}{a}}\sin(\frac{2\pi x}{a})$ for $|x| < a/2$ and zero elsewhere)?

i) One measures, with the system prepared in the state $\Psi(x) = e^{i\varphi}/\sqrt{a}$, in which energy eigenstate the system is. What is the probability for the measurement outcome that the system is in the ground state?

Problem 2

A certain atom is in a state with its total orbital angular momentum vector $\boldsymbol{L}$ (described by the operator $\hat{L}$) defined by orbital quantum number l = 1. For the system in this state, the operator for the z-component of angular momentum is $\hat{L}_z$. It has three eigenvalues, $+\hbar$ (with corresponding eigenstate $|+_z\rangle$), $0\hbar$ (with eigenstate $|0_z\rangle$), and $-\hbar$ (with eigenstate $|-_z\rangle$). This operator can be represented as a matrix, and the ket-states as column vectors, using the basis spanned by $|+_z\rangle$, $|0_z\rangle$ and $|-_z\rangle$, according to

$$\hat{L}_z \leftrightarrow \hbar\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad |+_z\rangle \leftrightarrow \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad |0_z\rangle \leftrightarrow \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \text{ and } \quad |-_z\rangle \leftrightarrow \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Using this same basis for the representation, the operator and eigenstates for the system's x-component of angular momentum are given by

$$\hat{L}_x \leftrightarrow \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad |+_x\rangle \leftrightarrow \begin{pmatrix} \frac{1}{2} \\ \frac{1}{\sqrt{2}} \\ \frac{1}{2} \end{pmatrix}, \quad |0_x\rangle \leftrightarrow \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{pmatrix}, \text{ and } \quad |-_x\rangle \leftrightarrow \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{\sqrt{2}} \\ \frac{1}{2} \end{pmatrix}.$$

a) Use this information to *calculate* what the eigenvalues are that belong to $|+_x\rangle$, $|0_x\rangle$ and $|-_x\rangle$.

b) At some point the system is in the normalized state $|\Psi_1\rangle = \sqrt{\frac{1}{3}}\,|+_z\rangle + \sqrt{\frac{1}{3}}\,|0_z\rangle + i\,\sqrt{\frac{1}{3}}\,|-_z\rangle$. Calculate for this state the expectation value for angular momentum in z-direction and the expectation value for angular momentum in x-direction.

c) Calculate for this state $|\Psi_1\rangle$ the quantum uncertainty ΔL_z in the z-component of the system's angular momentum.

d) With the system still in this same state $|\Psi_1\rangle$, you are going to measure the x-component of the system's angular momentum. What are the possible measurement results? Calculate the probability for getting the measurement result with the highest value for angular momentum in x-direction.

e) Now the system is prepared in a different state (now superposition of eigenstates of $\hat{L}_x$), $|\Psi_2\rangle = \sqrt{\frac{1}{2}}\,|+_x\rangle - \sqrt{\frac{1}{2}}\,|-_x\rangle$. You are going to measure the z-component of the system's angular momentum. What is the probability to find the answer $+\hbar$?

f) Now the system is prepared in a different state (now again a superposition of eigenstates of $\hat{L}_z$), $|\Psi_3\rangle = \sqrt{\frac{1}{2}}\,|+_z\rangle + \sqrt{\frac{1}{2}}\,|0_z\rangle$, at time $t = 0$. Another change to the system is that one now applied an external magnetic field with magnitude B along the z-axis. The Hamiltonian of the system is now, $\hat{H} = \gamma B \hat{L}_z$, where γ is a constant that reflects how much the energy of angular momentum states shifts when applying the field. Calculate how the expectation value for angular momentum in x-direction depends on time.

Use Dirac notation and the operator $\hat{U} = e^{\frac{-i\hat{H}t}{\hbar}}$.

Problem 3

a) Consider a quantum harmonic oscillator system that consists of a particle with mass m and charge q, that can move along x-direction in a potential $V_1(x) = m\omega^2x^2/2$. This entire system is placed in an instrument where an electrical field $\boldsymbol{E}$ along the x-direction (with magnitude E) can be switched on. With the field on, the particle feels the potential $V_2(x)$,

$$V_2(x) = m\omega^2 x^2/2 - qE\,x\ .$$

Discuss how the following properties change when the field is switched on:
a1) How do the energy eigenfunctions change?
a2) How do the energy eigenvalues change?
a3) For the system in the ground state, how does the expectation value $\langle x\rangle$ change?

Check the next page for a few hints!

Hint 1: The potential $V_2(x)$ can be transformed into a harmonic oscillator potential of the form $V_2(x) = V_0 + C(x - x_0)^2$. First calculate V_0, C and x_0. Then answer the questions.
Hint 2: For **a3)** you don't need to give a full calculation of $\langle x \rangle$ to answer the question.

b) Consider again the system as in **a)**, with $V_1(x) = m\omega^2x^2/2$. Now the system is placed in a different instrument, that changes the potential into V_3(x),

$$V_3(x) = \begin{cases} m\omega^2 x^2/2 & (x \geq 0) \\ \infty & (x < 0) \end{cases} .$$

Note that for $V_1(x)$ the system has the energy eigenfunctions $\varphi_n = A_n H_n(\xi)e^{-\xi^2/2}$ and the energy eigenvalues $E_n = \hbar\omega_0(n + 1/2)$, where $\xi = \sqrt{\frac{m\omega_0}{\hbar}}x$, A_n are constants to normalize φ_n, $H_n(\xi)$ are *Hermite polynomials* (*n*th-order polynomials) and $n = 1,2,3,\ldots$.

b1) Sketch $V_3(x)$ and explain for the energy eigenstates for the system with $V_3(x)$ what the conditions are for $x < 0$ and what the boundary conditions are at $x = 0$.

b2) The mentioned energy eigenvalues E_n (above question b1) are solutions of the time-independent Schrödinger equation for the system with $V_1(x)$. Write down the time-independent Schrödinger equation for this system in as much detail as you can (all as a function of x). Explain qualitatively why the set of solutions of this problem with eigenvalues E_n is discrete (instead of a continuum of E_n values that are a solution to the problem).

b3) Now consider all solutions for the time-independent Schrödinger equation for the system with $V_3(x)$. Can you figure out, by reasoning, what the energy eigenstates and energy eigenvalues are for the system with $V_3(x)$? If yes, give a summary of these in terms of (or in comparison with) the energy eigenstates and energy eigenvalues for the system with $V_1(x)$. ***Hint:*** consider which discrete set of solutions (as also considered in question **b1)** and **b2)**) can exist for this system.

Uitwerking Final Exam
Kwantum fysica 1, 21 JAN 2010

Problem 1

a) $\bar{\Psi}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \psi(x)\, e^{-ikx} dx = \int_{-a/2}^{a/2} \frac{1}{\sqrt{2\pi}} \frac{e^{i\varphi}}{\sqrt{a}} e^{-ikx} dx$

$= \frac{1}{-ik} \frac{e^{i\varphi}}{\sqrt{2\pi a}} \left[e^{-ikx} \right]_{-a/2}^{a/2} = e^{i\varphi} \sqrt{\frac{a}{2\pi}} \underbrace{\frac{\sin\left(\frac{ka}{2}\right)}{\left(\frac{ka}{2}\right)}}_{\text{sinc function}}$

b) v is proportional to k, since $mv = p_x = \hbar k$

$\bar{\Psi}(k)$ of a) is a symmetric wavefunction around $k=0$, so $\langle k \rangle = \int_{-\infty}^{\infty} \bar{\Psi}(k)^* k \bar{\Psi}(k)\, dk = 0$. So, also $\langle v \rangle$ must be zero $\Rightarrow$ $\langle v \rangle = 0$.

This must be the case since the particle is trapped in the box, on average it is at a fixed location.

c) I Simple very rough estimate

For a state confined in a box, one should expect $\Delta x\, \Delta p_x \approx \frac{\hbar}{2} \Rightarrow$

So $\Delta x\, m \Delta v_{Heis} \approx \frac{\hbar}{2} \Rightarrow \Delta v_{Heis} \approx \frac{\hbar}{2m\Delta x}$

$\left.\begin{aligned} \Delta x &\approx 0.5 \text{ nm} \\ m &= 2 \cdot 10^{-30} \text{ kg} \\ \hbar &= 1.055 \cdot 10^{-34} \end{aligned}\right\} \Rightarrow \Delta v_{Heis} \approx 5.2 \cdot 10^{4} \text{ m/s}$

This is the Heisenberg uncertainty in the velocity, and about 50% (or 68%) of measurement results will fall in the interval

$\langle v \rangle - \Delta v_{Heis} < v_m < \langle v \rangle + \Delta v_{Heis}$

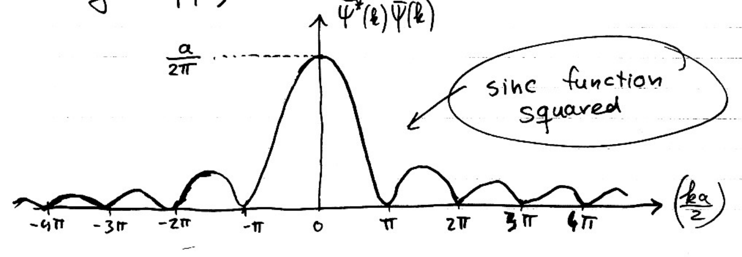
So, for a more-or-less Gaussian shape voor $|\bar{\Psi}(k)|^2$, one should use

$\Delta v_{90\%} \approx 2 \cdot \Delta v_{Heis} \Rightarrow$

$\boxed{\Delta v_{90\%} \approx 1 \cdot 10^{5} \text{ m/s}}$

II Little bit more quantitave rough estimate

As used for b), use that v is proportional to k, so the quantum properties of v can be analyzed using $\bar{\Psi}(k)$.



As sketched, $W(k) = |\bar{\Psi}(k)|^2 = \bar{\Psi}(k)^* \bar{\Psi}(k)$ 3/11

is a **sinc** function squared, and this probability density $W(k)$ determines measurement outcomes related to k.

For a rough estimate, use that

$$\lim_{k \to 0} W(k) = \frac{a}{2\pi} \lim_{k \to 0} \left(\frac{\sin(\frac{ka}{2})}{(\frac{ka}{2})} \right)^2 = \frac{a}{2\pi}$$

while for $\frac{ka}{2} \gg \pi$ the function $W(k)$ behaves on average as $W(k) \approx \frac{a}{2\pi} \left(\frac{1}{2} \frac{1}{(\frac{ka}{2})^2} \right)$

Then, using that $\langle k \rangle = 0$, it must be that

$$\int_{-\Delta k_{90\%}}^{+\Delta k_{90\%}} W(k)\, dk = 2 \int_0^{\Delta k_{90\%}} W(k)\, dk = 0.9 \Rightarrow$$

Since $\bar{\Psi}(k)$ is normalized, it must be that $2\int_0^{\infty} W(k)\, dk = 1 \Rightarrow$

$$2 \int_{\Delta k_{90\%}}^{\infty} W(k)\, dk = 0.1 \Rightarrow 2 \int_{\Delta k_{90\%}}^{\infty} \frac{a}{2\pi} \frac{1}{2} \frac{1}{(\frac{ka}{2})^2}\, dk \approx 0.1$$

$$\Rightarrow 2 \left[-\frac{1}{\pi a k} \right]_{\Delta k_{90\%}}^{\infty} \approx 0.1 \Rightarrow 2 \left[0 + \frac{1}{\pi a \Delta k_{90\%}} \right] = 0.1$$

$$\Rightarrow \Delta k_{90\%} = \frac{2}{0.1 \cdot \pi \cdot a}$$

$$\Delta v_{90\%} = \frac{\hbar \Delta k_{90\%}}{m} = \frac{\hbar \cdot 2}{0.1 \cdot \pi \cdot a} \approx 3 \cdot 10^5 \text{ m/s}$$

d) Use the time-independent Schrödinger equation in x-representation:

$$-\frac{\hbar^2}{2m} \frac{\partial^2 \varphi}{\partial x^2} = E \varphi$$

Filling in $\Psi(x)$ for φ with $\Psi(x) = \frac{e^{i\varphi}}{\sqrt{a}}$ for $|x| < \frac{a}{2}$ and $\frac{\partial^2 \Psi(x)}{\partial x^2} = 0$ gives

$$-\frac{\hbar^2}{2m} \cdot 0 = E \cdot \frac{e^{i\varphi}}{\sqrt{a}} \quad \text{for } |x| < \frac{a}{2}$$

This would hold if $E = 0$ would be an energy eigenvalue.
However, the Hamiltonian $H = \frac{\hat{p}^2}{2m} + V(x)$ only has terms ≥ 0, and the Heisenberg uncertainty relation forbids the state with both $\langle V(x) \rangle = 0$ and $\langle p \rangle = 0$.

So $\Psi(x)$ does not form a solution of the Schrödinger Eq.
So it is not an energy eigen state of the system.

e) $\langle x | \Psi \rangle$ equals the inner product $\int_{-\infty}^{\infty} \delta(x-x') \Psi(x')\, dx' = \Psi(x)$ by definition of the Dirac-delta function.

f) The system is in some quantum state, the physical state, which can be represented in many different ways.

$\Psi(x)$ represents this as a complex function (amplitude) as a function of x.

$|\Psi\rangle$ represents this as a state vector, which is an element of Hilbert space.
So, in a mathematical sense $\Psi(x)$ and $|\Psi\rangle$ cannot be equal.

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j) $\langle \varphi_n | \Psi \rangle = \langle \varphi_n | \left(\sum_{n'} c_{n'} |\varphi_{n'}\rangle \right)$

$= \sum_{n'} c_{n'} \langle \varphi_n | \varphi_{n'} \rangle$

For this system $\langle \varphi_n | \varphi_{n'} \rangle = \begin{cases} 0 & \text{for } n \neq n' \\ 1 & \text{for } n = n' \end{cases}$

So, $\langle \varphi_n | \Psi \rangle = c_n \langle \varphi_n | \varphi_n \rangle = c_n$

h) $c_1 = \langle \varphi_1 | \Psi \rangle = \int_{-\infty}^{\infty} \varphi_1(x) \Psi(x)\, dx$

8 $= \int_{-a/2}^{a/2} \sqrt{\frac{2}{a}} \cos\left(\frac{\pi x}{a}\right) e^{i\varphi} \frac{1}{\sqrt{a}}\, dx = e^{i\varphi} \frac{\sqrt{2}}{a} \cdot \frac{a}{\pi} \left[\sin\left(\frac{\pi x}{a}\right) \right]_{-a/2}^{a/2}$

$= e^{i\varphi} \frac{\sqrt{2}}{\pi} (1 - -1) = e^{i\varphi} \frac{2\sqrt{2}}{\pi}$

$c_2 = \langle \varphi_2 | \Psi \rangle = \int_{-\infty}^{\infty} \varphi_2(x) \Psi(x)\, dx$

$= \int_{-a/2}^{a/2} \sqrt{\frac{2}{a}} \sin\left(\frac{2\pi x}{a}\right) e^{i\varphi} \frac{1}{\sqrt{a}}\, dx = e^{i\varphi} \frac{\sqrt{2}}{2\pi} \left[-\cos\left(\frac{2\pi x}{a}\right) \right]_{-a/2}^{a/2} = 0$

(it must be zero since $\varphi_2(x) \cdot \Psi(x)$ is the product of an even and an odd function).

i) The ground state is $|\varphi_1\rangle$, so the probability is $|c_1|^2 = c_1^* c_1 = \frac{8}{\pi^2} \approx 0.81$

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Problem 2

a) The eigenvalues can be calculated with the eigenvalue equations for $\hat{L}_x$

For $|+_x\rangle$, $\frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{2} \\ \frac{1}{\sqrt{2}} \\ \frac{1}{2} \end{pmatrix} = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 1 \\ \frac{1}{\sqrt{2}} \end{pmatrix} = +\hbar \begin{pmatrix} \frac{1}{2} \\ \frac{1}{\sqrt{2}} \\ \frac{1}{2} \end{pmatrix} \Rightarrow \begin{cases} \text{its eigenvalue} \\ \text{is } +\hbar \end{cases}$

For $|0_x\rangle$, $\frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{pmatrix} = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = 0\hbar \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{pmatrix} \Rightarrow \begin{cases} \text{its eigenvalue} \\ \text{is } 0\hbar \\ (\hbar \text{ for unit}) \end{cases}$

For $|-_x\rangle$, $\frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{\sqrt{2}} \\ \frac{1}{2} \end{pmatrix} = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ 1 \\ -\frac{1}{\sqrt{2}} \end{pmatrix} = -\hbar \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{\sqrt{2}} \\ \frac{1}{2} \end{pmatrix} \Rightarrow \begin{cases} \text{its eigenvalue} \\ \text{is } -\hbar \end{cases}$

b) $\langle \hat{L}_z \rangle = \langle \Psi_1 | \hat{L}_z | \Psi_1 \rangle$

$= \left(\sqrt{\frac{1}{3}} \;\; \sqrt{\frac{1}{3}} \;\; -i\sqrt{\frac{1}{3}} \right) \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} \sqrt{\frac{1}{3}} \\ \sqrt{\frac{1}{3}} \\ +i\sqrt{\frac{1}{3}} \end{pmatrix}$

$= \left(\sqrt{\frac{1}{3}} \;\; \sqrt{\frac{1}{3}} \;\; -i\sqrt{\frac{1}{3}} \right) \hbar \begin{pmatrix} \sqrt{\frac{1}{3}} \\ 0 \\ -i\sqrt{\frac{1}{3}} \end{pmatrix} = \left(\frac{1}{3} + 0 - \frac{1}{3} \right) = 0\hbar$

($\hbar$ for unit)

$\langle \hat{L}_x \rangle = \langle \Psi_1 | \hat{L}_x | \Psi_1 \rangle$

$= \left(\sqrt{\frac{1}{3}} \;\; \sqrt{\frac{1}{3}} \;\; -i\sqrt{\frac{1}{3}} \right) \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \sqrt{\frac{1}{3}} \\ \sqrt{\frac{1}{3}} \\ +i\sqrt{\frac{1}{3}} \end{pmatrix} = \frac{\hbar}{\sqrt{2}} \left(\sqrt{\frac{1}{3}} \;\; \sqrt{\frac{1}{3}} \;\; -i\sqrt{\frac{1}{3}} \right) \begin{pmatrix} \sqrt{\frac{1}{3}} \\ \sqrt{\frac{1}{3}} + i\sqrt{\frac{1}{3}} \\ \sqrt{\frac{1}{3}} \end{pmatrix}$

$= \frac{\hbar}{\sqrt{2}} \left(\frac{1}{3} + \frac{1}{3} + i\frac{1}{3} - i\frac{1}{3} \right) = \frac{\hbar}{\sqrt{2}} \left(\frac{2}{3} \right) = \frac{2\hbar}{3\sqrt{2}} = \frac{1}{3}\sqrt{2}\,\hbar$

c) $\Delta L_z = \sqrt{\langle L_z^2\rangle - \langle L_z\rangle^2}$ (7/11)

So, we must first calculate the matrix representation of L_z^2

$$(L_z)^2 \leftrightarrow \begin{pmatrix}1&0&0\\0&0&0\\0&0&-1\end{pmatrix}\cdot\begin{pmatrix}1&0&0\\0&0&0\\0&0&-1\end{pmatrix}\hbar^2 = \hbar^2\begin{pmatrix}+1&0&0\\0&0&0\\0&0&+1\end{pmatrix}$$

Then, $\langle L_z^2\rangle = \langle \psi_1 | L_z^2 | \psi_1\rangle =$

$$\left(\sqrt{\tfrac{1}{3}}\ \ \sqrt{\tfrac{1}{3}}\ \ -\sqrt{\tfrac{1}{3}}\,i\right)\begin{pmatrix}1&0&0\\0&0&0\\0&0&1\end{pmatrix}\begin{pmatrix}\sqrt{1/3}\\ \sqrt{1/3}\\ +i\sqrt{1/3}\end{pmatrix}\hbar^2 = \left(\sqrt{\tfrac{1}{3}}\ \ \sqrt{\tfrac{1}{3}}\ \ -i\sqrt{\tfrac{1}{3}}\right)\begin{pmatrix}\sqrt{1/3}\\ 0\\ +i\sqrt{1/3}\end{pmatrix}\hbar^2$$

$$= \tfrac{1}{3}\hbar^2 + \tfrac{1}{3}\hbar^2 = \tfrac{2}{3}\hbar^2$$

Use from b) $\langle \hat{L}_z\rangle = 0$

$$\Delta L_z = \sqrt{\tfrac{2}{3}\hbar^2 - (0\hbar)^2} = \sqrt{\tfrac{2}{3}}\,\hbar$$

d) You measure L_x for $l=1$. From problem a) or general theory for L_x for $l=1$, the possible results are the eigenvalues of $\hat{L}_x$, which are $+\hbar$, $0\hbar$ and $-\hbar$.

$$P_{+\hbar} = |\langle +_x | \psi_1\rangle| = \left|\left(\tfrac{1}{2}\ \ \tfrac{1}{\sqrt{2}}\ \ \tfrac{i}{2}\right)\begin{pmatrix}\sqrt{1/3}\\ \sqrt{1/3}\\ i\sqrt{1/3}\end{pmatrix}\right|^2$$

$$= \left|\left(\tfrac{1}{2}\sqrt{\tfrac{1}{3}}\ \ \tfrac{1}{\sqrt{6}}\ \ i\tfrac{1}{2}\sqrt{\tfrac{1}{3}}\right)\right|^2 = \left(\tfrac{1}{6}\sqrt{3} + \tfrac{1}{6}\sqrt{6}\right)^2 + \tfrac{1}{12} \approx 0.57$$

e) $P_{+\hbar,z} = |\langle +_z | \psi_2\rangle|^2$ (8/11)

$$= \left|(1\ 0\ 0)\left(\tfrac{1}{\sqrt{2}}\begin{pmatrix}\tfrac{1}{2}\\ \tfrac{1}{\sqrt{2}}\\ \tfrac{1}{2}\end{pmatrix} - \tfrac{1}{\sqrt{2}}\begin{pmatrix}\tfrac{1}{2}\\ -\tfrac{1}{\sqrt{2}}\\ \tfrac{1}{2}\end{pmatrix}\right)\right|^2$$

$$= \left|(1\,0\,0)\cdot\tfrac{1}{\sqrt{2}}\begin{pmatrix}0\\ 2\sqrt{2}\\ 0\end{pmatrix}\right|^2 = 0$$

f) $\langle \hat{L}_x(t)\rangle = \langle \psi(t) | \hat{L}_z | \psi(t)\rangle$ with

$|\psi(t)\rangle = \hat{U}|\psi(0)\rangle = \hat{U}|\psi_3\rangle$

$$\hat{H} \leftrightarrow \begin{pmatrix}E_+&0&0\\0&E_0&0\\0&0&E_-\end{pmatrix} = \hbar\begin{pmatrix}\omega_+&0&0\\0&\omega_0&0\\0&0&\omega_-\end{pmatrix} \text{ with } \begin{cases}E_+ = +\gamma B\hbar\\ E_0 = 0\\ E_- = -\gamma B\hbar\\ \omega_+ = +\gamma B\\ \omega_0 = 0\\ \omega_- = -\gamma B\end{cases}$$

$$\Rightarrow |\psi(t)\rangle = \frac{e^{-i\omega_+ t}}{\sqrt{2}}|+_z\rangle + \frac{e^{-i\omega_0 t}}{\sqrt{2}}|0_z\rangle$$

$$\Rightarrow \langle L_x(t)\rangle = \left(\frac{e^{+i\omega_+ t}}{\sqrt{2}},\ \frac{e^{+i\omega_0 t}}{\sqrt{2}},\ 0\right)\begin{pmatrix}0&1&0\\1&0&1\\0&1&0\end{pmatrix}\begin{pmatrix}\frac{e^{-i\omega_+ t}}{\sqrt{2}}\\ \frac{e^{-i\omega_0 t}}{\sqrt{2}}\\ 0\end{pmatrix}\frac{\hbar}{\sqrt{2}}$$

$$= \frac{\hbar}{\sqrt{2}}\left(\frac{1}{2}\left(e^{+i(\omega_+-\omega_0)t} + e^{+i(\omega_0-\omega_+)t}\right)\right)$$

$$= \frac{\hbar}{\sqrt{2}}\cos\left((\omega_+ - \omega_0)t\right) = \underline{\underline{\frac{\hbar}{\sqrt{2}}\cos(\gamma B t)}}$$

Problem 3

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a) $V_2(x)$ can be written as

$$V_2(x) = V_0 + C(x-x_0)^2 \quad \text{with}$$

$$V_0 = -\frac{q^2E^2}{2m\omega^2}, \quad C = \frac{m\omega^2}{2}, \quad x_0 = \frac{qE}{m\omega}$$

$\Rightarrow$ $V_2(x)$ is the same harmonic potential as $V_1(x)$, but shifted in Energy by V_0 and in position by $+x_0$

a1) So, the eigenfunctions are $\tilde{\varphi}_n = \varphi_n(x-x_0)$

a2) The eigen values are $\tilde{E}_n = V_0 + (n+\frac{1}{2})\hbar\omega$

a3) As the eigenfunctions are shifted by $+x_0$ and $\langle x\rangle=0$ for the system with $V_1(x)$, $\langle x\rangle$ for the system with $V_2(x)$ must be $\langle x\rangle = x_0$.

b1)

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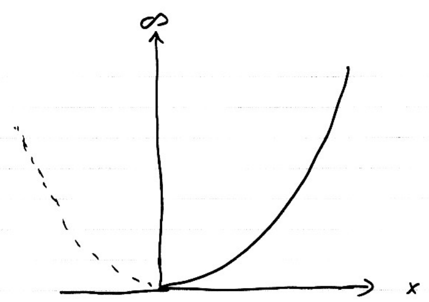

Condition for $x<0$: $\varphi_n(x) = 0$ for $x<0$.

Boundary condition at $x=0$: $\varphi_n(0) = 0$.

b2) Time indep. Schröd eq. $H(x)\,\varphi_n(x) = E_n\,\varphi_n(x) \Rightarrow$

$$-\frac{\hbar^2}{2m}\frac{\partial^2\varphi_n(x)}{\partial x^2} + V_1(x)\,\varphi_n(x) = E_n\,\varphi_n(x) \Rightarrow$$

$$-\frac{\hbar^2}{2m}\frac{\partial^2\varphi_n(x)}{\partial x^2} + \frac{m\omega^2x^2}{2}\varphi_n(x) = E_n\,\varphi_n(x).$$

There is only a discrete set of values E_n for which this problem has a solution, because the character of the solutions is that it are standing waves inside a potential that has bound solutions. Only states that constructively interfere with itself can exist, so each next solution requires a discrete step up for the number of $\frac{1}{2}$-wavelength that are present in the solutions.

b3) The time independent Schrödinger equation for $x > 0$ is exactly the same for $V_1(x)$ and $V_3(x)$. Therefore, the sequence with discrete solutions must also be similar. However, for $V_3(x)$ there is also the requirement that $\varphi_n(0) = 0$ (at $x=0$). Therefore, the system with $V_3(x)$ has these (and only these) energy eigenstates

$$\varphi_n(x) = \begin{cases} 0 & \text{for } x = 0 \\ A_n H_n(\xi) e^{-\xi^2/2}, & \text{for } x > 0 \end{cases}$$

and $\underline{\underline{n = 1, 3, 5, 7, 9, \ldots}}$

with $E_n = (n + \frac{1}{2})\hbar\omega$, with $n = 1, 3, 5, 7, 9, \ldots$

Note, $\varphi_n(x)$ for $n = 2, 4, 6, 8, \ldots$ has $\varphi_n(0) \neq 0$ at $x = 0$.